Transseries in difference and differential equations

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Chapter 1

Introduction

The problems studied in this thesis concern nonlinear analytic difference and differential equations in a neighbourhood of ∞ in the complex plane $\mathbb{C}$. In fact, we will study difference and differential systems of the form $y(x+1) = f(x, y(x))$ and $y'(x) + f(x, y(x)) = 0$ respectively, where $y = y(x) \in \mathbb{C}^n$. The nonlinear function $f(x, y) = f(x, y_1, y_2, \ldots, y_n)$ is assumed to have the form $f(x, y) = \Lambda(x)y + g(x, y)$, where Λ is an $n \times n$-matrix that is holomorphic for x in a neighbourhood of ∞ and where g is an n-vector valued function that is holomorphic for x in a neighbourhood of ∞ and y in a neighbourhood of 0 such that $g(x, y) = O(x^{-2}) + O(|y|^2)$ as $x \to \infty$ and $|y| \to 0$.

To fix the ideas, let us assume that $n = 1$ and that we are dealing with the equation

$$y(x+1) = e^{-\mu}(1 + x^{-1})^a y(x) + g(x, y(x)), \tag{1.0.1}$$

where $\mu \not\equiv 0 \bmod 2\pi i$ (i.e. $e^\mu \neq 1$) and $a \in \mathbb{C}$. Let $\mathcal{U}$ denote the algebra of formal expressions $\sum_{j=0}^{\infty} f_j(x)\, e^{-\sigma_j x}$ satisfying the following two properties.

(i) Every f_j can be written as $f_j = p_j g_j$ with $g_j \in \mathbb{C}((x^{-1}))[\{x^c\}_{c \in \mathbb{C}}]$ and p_j a 1-periodic $\mathbb{C}$-valued function of x;

(ii) Every σ_j belongs to $\mathbb{N} \cdot \mu$.

We note that the second condition implies that $\mathcal{U}$ indeed is an algebra. Moreover, if τ denotes the shift operator defined by $(\tau f)(x) := f(x+1)$, then τ acts naturally on $\mathcal{U}$ by

$$\tau\bigl(\textstyle\sum_{j=0}^{\infty} f_j(x) e^{-\sigma_j x}\bigr) = \textstyle\sum_{j=0}^{\infty} \bigl(\tau(f_j)(x)\, e^{-\sigma_j}\bigr) e^{-\sigma_j x} \quad \text{and} \quad \tau(f_j) = p_j\, \tau(g_j).$$

Here one should observe that if g_j equals $g_j(x) = a(x)x^c$, with $a \in \mathbb{C}((x^{-1}))$ and $c \in \mathbb{C}$, then $(\tau g_j)(x) = a(x+1)(1+x^{-1})^c x^c$ and $a(x+1)(1+x^{-1})^c$ belongs to $\mathbb{C}((x^{-1}))$. With the algebra $\mathcal{U}$ we associate the space

$$\mathrm{FSol}(\Delta) := \{y \in \mathcal{U} \mid y \text{ is a solution of } (1.0.1)\}.$$

Here Δ denotes the equation (1.0.1). The special form of (1.0.1) implies that there exists a unique formal solution $\hat{y}_0 \in x^{-1}\mathbb{C}[[x^{-1}]]$ of this equation and obviously this solution belongs to $\mathrm{FSol}(\Delta)$.

With the nonlinear equation (1.0.1) we associate the linear difference equation (in this thesis also referred to as the *normal form* corresponding to (1.0.1)) defined by

$$z(x+1) = e^{-\mu}(1+x^{-1})^a z(x). \tag{1.0.2}$$

We denote the space of its formal solutions in $\mathcal{U}$ by $\mathrm{FSol}(\Delta_{norm})$ and we will show that the general solution of (1.0.2) in $\mathcal{U}$ equals $z(x) = c(x)e^{-\mu x}x^a$, with c an arbitrary 1-periodic scalar function. Indeed, substituting $z(x) = c(x)e^{-\mu x}x^a$ in the normal form we obtain $c(x+1) = c(x)$. Hence,

$$\mathrm{FSol}(\Delta_{norm}) = \{c(x)e^{-\mu x}x^a \mid \text{for all } c \text{ with } c(x+1) = c(x)\}.$$

The general solution in the class of holomorphic functions on the Riemann surface of the logarithm is $z(x) = c(x)e^{-\mu x}x^a$, with c an arbitrary 1-periodic scalar holomorphic function.

The aim is to construct a (at first formal) transformation $y = \hat{T}(x,z)$ that formally transforms the nonlinear difference equation into its normal form and we require the transformation to be of the form

$$y = \hat{T}(x,z) := \sum_{k=0}^{\infty} \hat{y}_k(x) z^k,$$

where $\hat{y}_0$ is the unique formal power series solution of the nonlinear equation (1.0.1) found above and where $\hat{y}_k$, $k \geqslant 1$, are certain unknown expressions that we require to be elements of $\mathbb{C}[[x^{-1}]]$. It turns out that the $\hat{y}_k$, $k \geqslant 1$, have to satisfy linear difference equations of the form $y(x+1) = a_k(x)y(x) + t_k(x)$, where a_k is a convergent power series in the 'variables' x^{-1} and $\hat{y}_0$ and where t_k is some polynomial expression in $\hat{y}_{k'}$ with $0 \leqslant k' < k$. These equations can (at least formally) be solved, and thus there does exist a formal transformation $\hat{T}$. In particular we obtain a map $\hat{T} : \mathrm{FSol}(\Delta_{norm}) \to \mathrm{FSol}(\Delta)$, transforming formal solutions of the normal form into formal solutions of the original difference equation. Now, when we substitute a general element of $\mathrm{FSol}(\Delta_{norm})$ into the formal transformation one obtains the following *formal integral* of the difference equation

$$\hat{y}(x) = \sum_{k=0}^{\infty} c^k(x) e^{-k\mu x} x^{ka} \hat{y}_k(x) \in \mathrm{FSol}(\Delta) \tag{1.0.3}$$

(cf. [Eca85, CNP93]). The right-hand side of (1.0.3) belongs to the class of so-called *transseries* (cf. [Eca92]).

The next problem is to associate a holomorphic transformation with the given formal transformation $\hat{T}$ on a suitable sector S_1, in such a way that this holomorphic transformation has $\hat{T}$ as asymptotic expansion in a certain sense. For this we first need to associate holomorphic functions y_k with the constructed formal expressions $\hat{y}_k$ on the same sector S_1, having $\hat{y}_k$ as asymptotic expansion on S_1. If we succeed in this, then the series $\sum_{k=0}^{\infty} y_k(x)z^k$ turns out to be a convergent power series in z for every value of x in a neighbourhood U of ∞ in S_1. In other words: $T(x,z) := \sum_{k=0}^{\infty} y_k(x)z^k$ is a holomorphic transformation on $U \times \Delta(0,\rho)$ for some $\rho > 0$.

The formal power series $\hat{y}_k$, $k \in \mathbb{N}$, that occur here are Borel summable. This Borel summation method can be explained as follows. First one applies the formal Borel transform $\hat{\mathcal{B}}$ to the formal series $\hat{y}_k$. The resulting formal power series $\hat{\mathcal{B}}\hat{y}_k$ converges in a neighbourhood of 0 and has an analytic continuation along every half line $[0, \infty e^{i\theta})$, except for countably many. The directions of those exceptional half lines are called singular directions and are due to one or more singularities of $\hat{\mathcal{B}}\hat{y}_k$ on this half line. If θ is not a singular direction and $\theta \neq \pm\pi/2$, then this analytic continuation is such that the Laplace transform of $\hat{\mathcal{B}}\hat{y}_k$ with integration along the half line with direction θ exists. This Laplace transform produces a unique 'asymptotic lift' y_k of $\hat{y}_k$ on a sector with opening larger than π and bisecting direction $-\theta$. This asymptotic lift is called the *Borel sum* of $\hat{y}_k$ in the direction $-\theta$ (in for example [CNP93] this is called the Borel sum in direction θ).

In our case the singular directions of the formal solution $\hat{y}_0$ are given by $\arg(\mu + 2l\pi i)$, $l \in \mathbb{Z}$, and using the equation for $\hat{y}_k$ one easily infers that the singular directions of $\hat{y}_k$ are given by $\arg((1-j)\mu + 2l\pi i)$ for every $j \in \{0, 2, 3, \dots, k\}$ and $l \in \mathbb{Z}$. In the case θ_- and θ_+ are two consecutive singular directions in the right half plane of the set of all $\hat{y}_k$, the Borel summation method gives Borel sums y_k of $\hat{y}_k$ that all are holomorphic in a neighbourhood of ∞ in the sector $S_1 := \{x \in \mathbb{C}^* \mid -\pi/2 - \theta_+ < \arg x < \pi/2 - \theta_-\}$. Note that S_1 contains the positive real axis. From this it can be shown that $\hat{T}$ can be lifted to a unique holomorphic transformation of the type described above.

However, this does not automatically ensure convergence of the expression obtained by replacing $\hat{y}_k$ by their Borel sums y_k in the right-hand side of (1.0.3). In the following we will look at the convergence of this expression on certain sectors S_1 containing the positive real axis. As already mentioned, the *transformation* T converges provided that $|z|$ is small enough. Hence, if $c \not\equiv 0$, the corresponding transseries converges provided that $z(x) = c(x)e^{-\mu x}x^a$ tends to 0 as $x \to \infty$ in S_1, which implies $\Re\mu > 0$. It turns out that given a solution y of (1.0.1) such that y is 'small' on some sub-sector S_2 of S_1 containing the positive real axis, then there exists a unique convergent transseries such that y equals the sum of this transseries on S_2.

In the more general case where we study a system of difference equations

$$y(x+1) = \Lambda(x)y(x) + g(x, y(x)), \tag{1.0.4}$$

with Λ an $n \times n$-matrix ($n > 1$) of the form $\Lambda(x) = A_0(1 + x^{-1})^{A_1}$ and A_0 in diagonal form, the algebra $\mathcal{U}$ has to be replaced by formal expressions $\sum_{j=0}^{\infty} f_j(x)\, e^{-\sigma_j x}$ satisfying the following two properties.

(i) Every f_j can be written as $f_j = p_j g_j$ with $g_j \in \mathbb{C}^n((x^{-1}))[\{x^c\}_{c\in\mathbb{C}}, \log x]$ and p_j a 1-periodic $\mathbb{C}$-valued function of x;

(ii) There exist nonzero complex numbers $\mu_1, \mu_2, \dots, \mu_n$, such that every σ_j belongs to $\mathbb{N} \cdot \mu_1 + \mathbb{N} \cdot \mu_2 + \dots + \mathbb{N} \cdot \mu_n$.

With the nonlinear system of difference equations (1.0.4) one associates the *normal form*

$$z(x+1) = \Lambda(x) z(x). \tag{1.0.5}$$

It may be shown that (1.0.5) is uniquely solvable in $\mathcal{U}^n$, and this formal solution can actually be replaced by a holomorphic solution by replacing the 1-periodic functions by holomorphic 1-periodic functions.

Like in the 1-dimensional case it is possible to construct a formal transformation of the form $y = \hat{T}(x, z) = \sum_{\mathbf{k} \in \mathbb{N}^n} \hat{y}_{\mathbf{k}}(x) z^{\mathbf{k}}$ that formally transforms (1.0.4) into the normal form (1.0.5). However, the convergence of the corresponding transseries depends on the position of the eigenvalues of $\Lambda(\infty) = A_0$. Therefore it is assumed that the eigenvalues of A_0, which for convenience will be denoted by $e^{-\mu_j}$, $j = 1, 2, \dots, n$, are ordered in such a way that $\Re\mu_j > 0$ for $j = 1, 2, \dots, p$ and $\Re\mu_j \leqslant 0$ for $j \in \{p+1, p+2, \dots, n\}$. In the case that both A_0 and A_1 are in diagonal form, it has been shown by Braaksma in [Bra01] that only the 'partial' formal transformation

$$\hat{T}_1(x, u_1, \dots, u_p) := \hat{T}(x, u_1, \dots, u_p, 0, \dots, 0)$$

can be lifted to a holomorphic expression T_1: the formal series $\hat{T}_1(x, u)$ turns out to be Borel summable with respect to x in some sector S_1 containing the positive real axis, uniformly in u (provided that $|u|$ is small enough). Consequently, the Borel sum $T_1(x, u)$ of $\hat{T}_1(x, u)$ exists for x in a neighbourhood of ∞ in S_1 and u in a neighbourhood of 0 and the original difference equation, restricted to the manifold defined by $y = T_1(x, u)$, transforms into the so-called *semi-canonical form* $u(x+1) = \Lambda(x) u(x)$. However, the main result in [Bra01] is that given a solution y of (1.0.4) and given a sector $S_2 \subset S_1$ containing the positive real axis such that y is 'small' on S_2, there exists a unique convergent transseries such that y equals the sum of this transseries.

Similar results have been obtained by Costin in [Cos95, Cos98] for the analogous nonlinear rank 1 differential equation

$$y'(x) + \Lambda(x) y(x) + g(x, y(x)) = 0, \tag{1.0.6}$$

under the assumption that $\Lambda(x) = A_0 - x^{-1} A_1$, with both A_0 and A_1 diagonal matrices. Such systems have been studied also by Malmquist ([Malm40, Malm41]) and Iwano ([Iwa57, Iwa59]), compare also [Was87], chapter IX. Costin derived convergent transseries representations for 'small' solutions on sectors, resurgence relations and balanced averages. An important notion in his work is that of staircase distributions. In [CC01] Costin and Costin showed that at the boundary of a maximal sector where a solution of the system considered in [Cos98] is 'small', singularities of this solution occur that are situated in nearly periodic arrays.

One part of this thesis (chapters 4 and 5) is concerned with generalisations of the assumptions on Λ for both differential and difference equations: we derive similar results concerning the analytic reduction to a semi-canonical form for these generalised equations. In chapter 2 we extend Braaksma's study of the difference equation (1.0.4), while in chapter 3 we will extend the results in [CC01] to systems of difference equations.

Very general results concerning nonlinear meromorphic differential and difference equations of the type considered in this thesis have been formulated by Écalle (cf. [Eca85, Eca92]). His treatment includes the study of formal integrals, resurgence relations, alien derivations, the associated bridge equation, holomorphic invariants, accelero-summation and medianisation. An instructional treatment of Écalle's work for scalar differential equations has been given by Candelpergher, Nosmas and Pham ([CNP93]). They consider the construction of the formal integral and its resurgence properties, alien derivations, the bridge equation, Stokes transition and analytic classification.

A more detailed outline of the thesis is given in the next section. In section 1.2 we introduce some notations and terminology, while in section 1.3 and section 1.4 the notion of Borel summability and multisummability will be discussed. In section 1.5 we study in detail two examples that might clarify some of the above.

1.1 General Outline of the Thesis

In chapter 2 we consider difference equations of only one level, namely *level one*. This means that the matrix $\Lambda(x) = A_0(1 + x^{-1})^{A_1}$ has the form $\Lambda(x) = \oplus_{j=1}^{r} e^{-\mu_j}(1 + x^{-1})^{\mathtt{M}_j}$, with $\mu_j \not\equiv 0 \bmod 2\pi i$, and we assume that each $\mathtt{M}_j$ is a diagonal matrix with complex numbers on the diagonal. In fact, this chapter contains results for difference equations analogous to the results Costin gave in [Cos95, Cos98] for rank 1 differential equations. As the existence and unicity of the transformation $\hat{T}$ and the asymptotic lift to T_1 both are discussed by Braaksma in [Bra01], these results will only be summarised. The main focus of this chapter will be on the behaviour of the Borel transform of $\hat{y}_{\mathbf{k}}$ near and on singular rays. In this study we use the theory of Costin's staircase distributions ([Cos98]). The convolution equations on singular rays, solutions on singular rays, resurgence relations, Stokes transition and balanced averages will be discussed.

Chapter 3 contains results about difference equations analogous to the results Costin and Costin obtained in [CC01] for differential equations. We consider a special type of the class of equations that we studied in chapter 2, namely those with Λ of the form $\Lambda(x) = \mathrm{diag}\{e^{-\mu_1}(1+x^{-1})^{a_1}, \ldots, e^{-\mu_n}(1+x^{-1})^{a_n}\}$. The actual transseries solution will be studied in more detail. A first result is that the solution y can be extended to a larger region than the one obtained in chapter 2. Loosely speaking this region cannot be enlarged in the sense that singularities occur near the border of this extended region. It turns out that the singular points are grouped together in nearly periodic arrays.

In chapter 4 we will generalise the results in [Bra01] concerning analytic reduction to a semi-canonical form to that case where we admit nilpotent matrices in the linear part of the equation we take into consideration. In this chapter we will study both differential and difference equations. In the differential case we take $\Lambda(x) = \oplus_{j=1}^{r}(\mu_j \mathtt{I}_{n_j} - x^{-1}\mathtt{M}_j)$, while in the case of difference equations we take $\Lambda(x) = \oplus_{j=1}^{r} e^{-\mu_j}(1+x^{-1})^{\mathtt{M}_j}$. For each $j \in \{1, 2, \ldots, r\}$ we only assume that two eigenvalues of $\mathtt{M}_j$ do not differ by a nonzero integer, which makes it possible to compute formal solutions of the equations for $\hat{y}_{\mathbf{k}}$. Apart from this assumption this is the most general form for the matrices $\mathtt{M}_j$.

The last chapter deals with differential equations $y'(x) + \Lambda(x)y(x) + g(x, y(x)) = 0$ in the case that Λ has a pole at $x = \infty$. Formal solutions of such equations are not Borel summable, but in [Bra92, RS94, Bal94] it is shown that such formal solutions are multisummable. We first consider differential equations as above with three levels. In fact, we will first restrict ourselves to equations with the coefficient Λ of the form

$$\begin{aligned} \Lambda(x) = \; & \operatorname{diag}\{\omega_m x^{q_1-1} + \lambda_m x^{q_2-1} + \mu_m x^{q_3-1} - a_m x^{-1}\}_{m=1}^{n_1} \\ & \oplus \operatorname{diag}\{\lambda_m x^{q_2-1} + \mu_m x^{q_3-1} - a_m x^{-1}\}_{m=n_1+1}^{n_1+n_2} \\ & \oplus \operatorname{diag}\{\mu_m x^{q_3-1} - a_m x^{-1}\}_{m=n_1+n_2+1}^{n}, \end{aligned}$$

where $0 < q_3 < q_2 < q_1 < \infty$, $n = n_1 + n_2 + n_3$, $q_j, n_j \in \mathbb{N}$, $j = 1, 2, 3$. In section 5.6 we will give a generalisation to r levels, with $r \in \mathbb{N}_+$.

In appendix A we discuss the theory on staircase distributions, first introduced by Costin in [Cos98]. Most of the results we prove in this appendix can be found in Costin's article. In appendix B we give an existence theorem for a class of linear difference equations that will be used in chapter 3 and chapter 4.

1.2 Some Notations and Terminology

1.2.1 Regions in the Complex Plane

In this thesis an open disc with centre $x \in \mathbb{C}$ and radius $\rho > 0$ will be denoted by $\Delta(x, \rho)$, while its closure will be denoted by $\overline{\Delta}(x, \rho)$. A sector (on the Riemann surface of the logarithm) is defined to be a set of the form

$$S(\theta, \alpha) := \{x \in \mathbb{C}^* \mid \theta - \alpha/2 < \arg x < \theta + \alpha/2\},$$

where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and where $\theta \in \mathbb{R}$ and $\alpha > 0$. We shall refer to θ and α as the *bisecting direction* and the *opening* of $S(\theta, \alpha)$ respectively. Given a direction $\theta \in \mathbb{R}$, the half line $\{x \in \mathbb{C}^* \mid \arg x = \theta\}$ will also be denoted by its direction θ. A closed[1] sub-sector $\overline{S}_1 \subset S(\theta, \alpha)$ is a set of the form

$$\overline{S}_1 := \{x \in S(\theta, \alpha) \mid \beta_1 \leqslant \arg x \leqslant \beta_2\},$$

where $\theta - \alpha/2 < \beta_1 \leqslant \beta_2 < \theta + \alpha/2$. A neighbourhood of ∞ in a sector $S(\theta, \alpha)$ will be defined by

$$\big\{x \in S(\theta, \alpha) \bigm| |x| > r(\arg x)\big\},$$

[1]Here *closed* means closed with respect to the induced topology of $\mathbb{C}$ on $\mathbb{C}^*$.

where r is some positive valued continuous function on $(\theta - \alpha/2, \theta + \alpha/2)$. In some cases r can be taken a constant function, in other cases r is a function which tends to ∞ as $\arg x \to \theta \pm \alpha/2$. A neighbourhood of 0 in $S(\theta, \alpha)$ is defined similarly, but with $>$ replaced by $<$.

1.2.2 Asymptotics

Given a sequence $\{f_m\}_{m=0}^{\infty}$ of complex numbers, the series $\hat{f}(x) := \sum_{m=0}^{\infty} f_m x^{-m}$ is called a formal power series (in x^{-1}), the term 'formal' emphasising that we do not restrict the numbers f_m in any way. Thus the radius of convergence of the series $\sum_{m=0}^{\infty} f_m x^{-m}$ may well be equal to zero. The set of such formal power series is denoted by $\mathbb{C}[[x^{-1}]]$. Now, if f is holomorphic in a neighbourhood of ∞ in a sector S and if $\hat{f}(x) = \sum_{m=0}^{\infty} f_m x^{-m} \in \mathbb{C}[[x^{-1}]]$ is a formal series, we say that $f(x)$ asymptotically equals $\hat{f}(x)$ as $x \to \infty$ in S ($f(x) \sim \hat{f}(x)$ as $x \to \infty$ in S) if, for every nonnegative integer N and every closed sub-sector $\overline{S}_1$ of S, we have

$$f(x) = \sum_{m=0}^{N-1} f_m x^{-m} + O(x^{-N}), \quad \text{as } x \to \infty \text{ in } \overline{S}_1.$$

More precisely, for every nonnegative integer N and every closed sub-sector $\overline{S}_1$ there exists a positive constant C depending on both N and $\overline{S}_1$, such that

$$|f(x) - \textstyle\sum_{m=0}^{N-1} f_m x^{-m}| \leqslant C|x|^{-N}, \qquad \text{for all } x \in \overline{S}_1.$$

If the constant C can be chosen in the form $C = cN!K^N$, for some positive constants c and K independent of N, we say that $f(x)$ asymptotically equals $\hat{f}(x)$ of *Gevrey order 1*. In that case we write $f(x) \sim_1 \hat{f}(x)$ as $x \to \infty$ in S.

Similarly, given a function g holomorphic in a neighbourhood of 0 in a sector S and a formal series $\hat{g}(t) := \sum_{m=0}^{\infty} g_m t^m \in \mathbb{C}[[t]]$, we say that $g(t)$ asymptotically equals $\hat{g}(t)$ as $t \to 0$ in S ($g(t) \sim \hat{g}(t)$ as $t \to 0$ in S) if, for every nonnegative integer N and every closed bounded sub-sector $\overline{S}_1$ of S, we have $g(t) = \sum_{m=0}^{N-1} g_m t^m + O(t^N)$ as $t \to 0$ in $\overline{S}_1$. If the difference $g(t) - \sum_{m=0}^{N-1} g_m t^m$ can be estimated by $cN!K^N|t|^N$ for $t \in \overline{S}_1$, then $g(t)$ asymptotically equals $\hat{g}(t)$ of Gevrey order 1 and this is expressed as $g(t) \sim_1 \hat{g}(t)$ as $t \to 0$ in S.

1.2.3 Multi-indices

The notation $\mathbb{N}$ will be used for the set of natural numbers *including* 0. The notation $\mathbb{N}_+$ will be reserved for the set $\{1, 2, 3, \ldots\}$. Elements of the set $\mathbb{N}^n$, with $n \in \mathbb{N}_+$, will be called multi-indices. These multi-indices are ordered in the following way: if both $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ and $\mathbf{l} = (l_1, l_2, \ldots, l_n)$ are elements of $\mathbb{N}^n$, then we will write $\mathbf{k} \preceq \mathbf{l}$ if $k_j \leqslant l_j$ for every $j \in \{1, 2, \ldots, n\}$. If moreover $\mathbf{k} \neq \mathbf{l}$ (meaning $k_j \neq l_j$ for some j), then we will use the notation $\mathbf{k} \prec \mathbf{l}$. In these cases we will also write $\mathbf{l} \succeq \mathbf{k}$ and $\mathbf{l} \succ \mathbf{k}$ respectively.

The j^{th} unit vector in $\mathbb{C}^n$ will be denoted by $\mathbf{e}_j$. For a multi-index $\mathbf{k} \in \mathbb{N}^n$ the notation $|\mathbf{k}|$ (also referred to as the *length* of $\mathbf{k}$) will be used for the sum of the components of $\mathbf{k}$, i.e.

$|\mathbf{k}| = \sum_{j=1}^{n} k_j$. For $r \in \mathbb{N}_+$ we define $\mathbb{N}_r^n := \{\mathbf{k} \in \mathbb{N}^n \mid |\mathbf{k}| \geqslant r\}$. For two multi-indices $\mathbf{k}$ and $\mathbf{l}$, the notation $\binom{\mathbf{k}}{\mathbf{l}}$ will be used for the expression $\prod_{j=1}^{n} \binom{k_j}{l_j}$. If $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, then $z^{\mathbf{k}}$ is defined by $z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$. Moreover, the map $\langle \cdot, \cdot \rangle : \mathbb{N}^n \times \mathbb{C}^n \to \mathbb{C}$ is defined by the bilinear form $\langle \mathbf{k}, z \rangle := \sum_{j=1}^{n} k_j z_j$.

1.3 Borel Summability

In this section we will give a concise review of the notion of Borel summability. For a detailed exposition of this theory we refer to the papers [MR91, MalR92, Mal95]. In [Bal94, Bal00] Balser also gives a complete and detailed overview of Borel summability. However, he uses slightly different definitions from those we will give in this thesis.

1.3.1 The Borel Transform

Let f be a holomorphic function in a neighbourhood U of ∞ in the sector $S(-\theta, \alpha + \pi)$, where $\alpha > 0$, and assume that $f(x) = O(x^{\delta})$ as $x \to \infty$ in U, where $\delta \in \mathbb{R}$. Then the Borel transform of f is defined by

$$(\mathcal{B}_{\gamma} f)(t) := \frac{1}{2\pi i} \int_{\gamma} f(x) e^{tx} \mathrm{d}x, \tag{1.3.1}$$

for $\pi/2 - \theta_2 < \arg t < -\pi/2 - \theta_1$ if γ is an unbounded contour in U with two limiting directions θ_1 and θ_2 with $\theta_2 - \theta_1 > \pi$. See figure 1 (in which the shaded part denotes the sector $S(-\theta, \alpha+\pi)$). By deformation of γ, $\mathcal{B}_{\gamma} f$ can be continued analytically to $S(\theta, \alpha)$. If

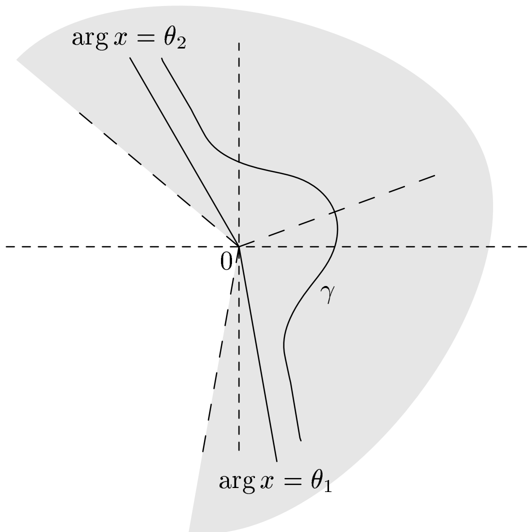

Figure 1.1: *The contour γ.*

we denote $\mathcal{B} f$ to be the Borel transform of f in $S(\theta, \alpha)$, then $\mathcal{B} f$ is of at most exponential growth in $S(\theta,\alpha)$, meaning that for any closed sub-sector $\overline{S}_1$ of $S(\theta,\alpha)$ there exists a positive constant M such that $\sup_{t\in\overline{S}_1} e^{-M|t|}|(\mathcal{B}f)(t)| < \infty$.

The formal Borel transform of a convergent or divergent series $\hat{f}(x) = \sum_{m=0}^{\infty} c_m x^{-m-r}$, with $r \in \mathbb{C}$, is defined by

$$(\hat{\mathcal{B}}\hat{f})(t) := \sum_{m=0}^{\infty} c_m \frac{t^{m+r-1}}{\Gamma(m+r)}. \tag{1.3.2}$$

Let $\hat{g}$ be a series of the same form as $\hat{f}$, but with r replaced by s. If $\hat{f}$ and $\hat{g}$ have (formal) Borel transforms F and G, $\Re r > 0$ or $r = 0$ and $\Re s > 0$ or $s = 0$, and $t \mapsto t^{1-r}F(t)$ and $t \mapsto t^{1-s}G(t)$ are convergent in some disc $\Delta(0,\rho)$, $\rho > 0$, then

$$\hat{\mathcal{B}}(\hat{f}\hat{g}) = \hat{f}(\infty)G + \hat{g}(\infty)F + F * G,$$

where $(F * G)(t) := \int_0^t F(t-\sigma)G(\sigma)\,\mathrm{d}\sigma$ for all $t \in \Delta(0,\rho)$. Moreover, if $r = 1$ and $f(x) \sim \hat{f}(x)$ as $x \to \infty$ in $S(-\theta,\alpha+\pi)$, then $(\mathcal{B}f)(t) \sim (\hat{\mathcal{B}}\hat{f})(t)$ as $t \to 0$ in $S(\theta,\alpha)$.

1.3.2 The Laplace Transform

Let F be a holomorphic function of at most exponential growth in $S(\theta,\alpha)$. Moreover, assume that $F(t) = O(t^{\varepsilon-1})$ as $t \to 0$ in $S(\theta,\alpha)$ for some $\varepsilon > 0$. Then the Laplace transform of f in the direction $\psi \in (\theta - \alpha/2, \theta + \alpha/2)$ is defined by

$$(\mathcal{L}_\psi F)(x) := \int_0^{\infty e^{i\psi}} F(t)\, e^{-xt}\,\mathrm{d}t, \tag{1.3.3}$$

with integration along $\arg t = \psi$. It is easily seen that this integral converges for $\Re(xe^{i\psi})$ sufficiently large, thus $\mathcal{L}_\psi F$ is holomorphic in a neighbourhood of ∞ in $S(-\psi,\pi)$. Moreover, it is easy to see that varying $\psi \in (\theta - \alpha/2, \theta + \alpha/2)$ gives rise to analytic continuations of this Laplace transform. So we end up with a Laplace transform $\mathcal{L}F$ that is holomorphic in a neighbourhood of ∞ in $S(-\theta,\alpha+\pi)$.

Given a convergent or divergent series $\hat{F}(t) = \sum_{m=0}^{\infty} c_m t^m \in \mathbb{C}[[t]]$, the formal Laplace transform of $\hat{F}$ is defined by

$$(\hat{\mathcal{L}}\hat{F})(x) := \sum_{m=0}^{\infty} c_m \Gamma(m+1)\, x^{-m-1} \in x^{-1}\mathbb{C}[[x^{-1}]]. \tag{1.3.4}$$

If $F(t) \sim \hat{F}(t)$ as $t \to 0$ in $S(\theta,\alpha)$, and the above assumptions on F are satisfied, then $(\mathcal{L}F)(x) \sim (\hat{\mathcal{L}}\hat{F})(x)$ as $x \to \infty$ in $S(-\theta,\alpha+\pi)$. In that case we have $(\mathcal{B}\mathcal{L})F = F$. On the other hand, if f is holomorphic in a neighbourhood of ∞ in a sector $S(-\theta,\alpha+\pi)$ and $f(x) \sim \sum_{m=0}^{\infty} c_m x^{-m-1}$ as $x \to \infty$ in $S(-\theta,\alpha+\pi)$, then $(\mathcal{L}\mathcal{B})f = f$.

If F and G are holomorphic and of at most exponential growth in $S(\theta,\alpha)$ and if both are of order $O(t^{\varepsilon-1})$ as $t \to 0$ in $S(\theta,\alpha)$ for some $\varepsilon > 0$, then the convolution product of F and G is defined, holomorphic and of at most exponential growth in $S(\theta,\alpha)$. Moreover, $F * G$ is of order $O(t^{2\varepsilon-1})$ as $t \to 0$ in $S(d,\alpha)$. Hence, $\mathcal{L}(F * G)$ is defined and holomorphic in a neighbourhood of ∞ in $S(-\theta,\alpha+\pi)$. It is known that $\mathcal{L}(F * G) = \mathcal{L}F \cdot \mathcal{L}G$ in this neighbourhood.

1.3.3 Definition of Borel Summability

Given a formal series $\hat{f}(x) = \sum_{m=0}^{\infty} c_m x^{-m-1} \in x^{-1}\mathbb{C}[[x^{-1}]]$, we say that $\hat{f}$ is *Borel summable in the direction* $-\theta$ if $\hat{\mathcal{B}}\hat{f}$ converges in a neighbourhood of the origin, with sum F, and F can be analytically continued in a sector $S(\theta, \delta)$ for some small positive δ and has at most exponential growth in this sector. If $\hat{f}$ is Borel summable in the direction $-\theta$, then we call $f := \mathcal{L}F$ the *Borel sum* of $\hat{f}$. Note that this Borel sum is holomorphic in a neighbourhood of ∞ in $S := S(-\theta, \pi + \delta)$ and it asymptotically equals $\hat{f}$ as $x \to \infty$ in S. If $\hat{f} \in x^{-1}\mathbb{C}[[x^{-1}]]$ is Borel summable in all but countably many directions, we just call $\hat{f}$ Borel summable.

The set of Borel summable series is closed under addition, multiplication and differentiation, i.e. if both $\hat{f}$ and $\hat{g}$ are Borel summable (in the direction $-\theta$) with Borel sums f and g respectively, then also $\hat{f} + \hat{g}$, $\hat{f}\hat{g}$ and $\hat{f}'$ are Borel summable (in the direction $-\theta$) with sums $f + g$, fg and f' respectively. Moreover, if $\hat{f}$ is Borel summable with Borel sum f and $\hat{f}(\infty) \neq 0$, then $\hat{f}^{-1}$ is Borel summable with Borel sum f^{-1}. If $\hat{f} \in \mathbb{C}\{x^{-1}\}$ is some convergent series in x^{-1}, then $\hat{f}$ is Borel summable in every direction $-\theta$ and its Borel sum equals the sum of the convergent series.

A formal series $\hat{f}(x) := \sum_{m=0}^{\infty} c_m x^{-m}$ is Borel summable if $\hat{g}(x) := \sum_{m=1}^{\infty} c_m x^{-m}$ is Borel summable and in that case the Borel sum of $\hat{f}$ is defined by $c_0 + g$, where g is the Borel sum of $\hat{g}$. If $\hat{f}(x) = x^{-r}\hat{g}(x)$, with $\Re r > 0$, and $\hat{g}(x) = \sum_{m=0}^{\infty} c_m x^{-m}$, then $\hat{f}$ is called Borel summable if $\hat{g}$ is Borel summable and its Borel sum equals $f(x) = x^{-r}g(x)$, where g is the Borel sum of $\hat{g}$. It is not hard to prove that this definition is equivalent to the following: the series $\hat{f}(x) := \sum_{m=0}^{\infty} c_m x^{-m-r}$ is Borel summable if $t \mapsto t^{1-r}(\hat{\mathcal{B}}\hat{f})(t)$ converges in a neighbourhood of the origin and the sum F of the series $\hat{\mathcal{B}}\hat{f}$ can be analytically continued in a certain sector and is of at most exponential growth in that sector. If so, then its Borel sum equals $f := \mathcal{L}F$.

Example 1.3.1 (Euler's equation) The differential equation $y'(x) - y(x) + x^{-1} = 0$ is known as *Euler's equation*[2]. A formal substitution of a series $\sum_{m=0}^{\infty} \alpha_m x^{-m}$ leads to[3]

$$-\sum_{m=2}^{\infty}(m-1)\alpha_{m-1}x^{-m} - \sum_{m=0}^{\infty} \alpha_m x^{-m} + x^{-1} = 0.$$

Comparing coefficients of x^{-m}, $m \geqslant 0$, gives $\alpha_0 = 0$, $\alpha_1 = 1$ and $\alpha_m = -(m-1)\alpha_{m-1}$ for $m \geqslant 2$, and thus $\alpha_0 = 0$ and $\alpha_m = (-1)^{m-1}(m-1)!$ for $m \in \{1, 2, 3, \ldots\}$. Hence, Euler's equations possesses a unique *formal series solution* $\hat{y}(x) := \sum_{m=0}^{\infty}(-1)^m m!\, x^{-m-1}$.

Taking a formal Borel transform of this formal solution gives the series $\sum_{m=0}^{\infty}(-1)^m t^m$, which converges in the disc $\Delta(0, 1)$. Moreover, its sum can be analytically extended as the function $t \mapsto (1+t)^{-1}$ in every sector S not containing the negative real axis. Obviously, the

[2]This is Euler's equation written down for x in a neighbourhood of ∞. The equivalent form near 0 looks like $t^2\tilde{y}'(t) + \tilde{y}(t) - t = 0$.

[3]We just substitute the series $\sum_{m=0}^{\infty} \alpha_m x^{-m}$ into the equation and interchange summation and differentiation without verifying whether it is allowed are not.

latter function is of at most exponential growth in such a sector, so $\hat{y}$ is Borel summable in every direction $-\theta$, with $\theta \neq \pi$ and its Borel sum equals the Laplace transform of $(1+t)^{-1}$ (where we can integrate along any half line starting at 0, except for the negative real axis).

By means of variation of constants one easily obtains the following general holomorphic solution of Euler's equation:

$$y(x) = c\, e^x + \int_x^\infty \frac{e^{x-\sigma}}{\sigma}\, \mathrm{d}\sigma, \quad c \in \mathbb{C}. \tag{1.3.5}$$

Substituting $\sigma = x(1+t)$ and taking $c = 0$ we see that $y(x) = \int_0^\infty e^{-xt} \frac{\mathrm{d}t}{1+t}$, which is exactly the Borel sum of $\hat{y}$. So, the Borel sum of $\hat{y}$ indeed is a holomorphic solution of Euler's equation.

Another way to reach the same conclusion is by transforming Euler's equation into a corresponding equation by applying the (formal) Borel transform. This gives

$$-tY - Y + 1 = 0,$$

where $Y := \hat{\mathcal{B}}\hat{y}$. The latter equation has the unique solution $Y(t) = \frac{1}{1+t}$. As the Borel and Laplace operator are each others inverse, the Laplace transform of Y is a (holomorphic) solution of Euler's equation. Moreover, from the theory above one immediately deduces that this holomorphic solution asymptotically equals $\sum_{m=0}^\infty (-1)^m m!\, x^{-m-1}$ as $x \to \infty$ in any sector not containing the directions $\pm\frac{\pi}{2}$ mod 2π. So the formal solution $\hat{y}$ is lifted to an actual solution of Euler's equation.

1.4 Multisummability

Again we only give a brief overview of the definitions of the generalised Laplace and Borel transforms, acceleration operators and the notion of multisummability, which are due to Écalle (cf. [Eca85, Eca90, Eca91, Eca92]). In this section we will follow the paper [MR91] of Martinet and Ramis. However, equivalent forms of multisummability have been given by Malgrange and Ramis in [MalR92, Mal95] and by Balser in [Bal92, Bal94, Bal00] (the latter with slightly different definitions).

In the remaining part of this chapter q will always denote a positive number.

1.4.1 The Borel Transform of Order q

Let f be a holomorphic function in a neighbourhood U of ∞ in the sector $S(-\theta, \alpha + \frac{\pi}{q})$, where $\alpha > 0$, and assume that $f(x) = O(x^\delta)$ as $x \to \infty$ in U, where $\delta \in \mathbb{R}$. Then the Borel transform of order q of f is defined by

$$(\mathcal{B}_q f)(t) := \frac{1}{2\pi i} \int_\gamma f(x)\, e^{(tx)^q}\, \mathrm{d}(x^q), \tag{1.4.1}$$

for a certain unbounded contour γ in U, which may depend on q. Here $\mathrm{d}(x^q)$ is a short-hand notation for $qx^{q-1}\mathrm{d}x$. The integration contour γ has the same form as the path of integration in the definition of the ordinary Borel transform (cf. figure 1.1), but now with limiting directions θ_1 and θ_2 with $\theta_2 - \theta_1 > \pi/q$ and then $\frac{\pi}{2q} - \theta_2 < \arg t < -\frac{\pi}{2q} - \theta_1$. Again changes of γ give rise to analytic continuations and by varying γ one obtains a Borel transform of order q of f, which is holomorphic and of exponential growth of order $\leqslant q$ in $S(\theta,\alpha)$, the latter meaning that for any closed sub-sector $\overline{S}_1$ of $S(\theta,\alpha)$ there exists a positive constant M such that $\sup_{t\in\overline{S}_1} e^{-M|t|^q}|(\mathcal{B}_q f)(t)| < \infty$.

Given a convergent or divergent series $\hat{f}(x) = \sum_{m=0}^{\infty} c_m x^{-m-1} \in x^{-1}\mathbb{C}[[x^{-1}]]$, its formal Borel transform of order q is defined by

$$(\hat{\mathcal{B}}_q \hat{f})(t) := \sum_{m=0}^{\infty} c_m \frac{t^{m+1-q}}{\Gamma(\frac{m+1}{q})}. \tag{1.4.2}$$

If $f(x) \sim \hat{f}(x)$ as $x \to \infty$ in $S(-\theta, \alpha + \frac{\pi}{q})$, then $(\mathcal{B}_q f)(t) \sim (\hat{\mathcal{B}}_q \hat{f})(t)$ as $t \to 0$ in $S(\theta,\alpha)$. Note that this has to be interpreted as $t^{q-1}(\mathcal{B}_q f)(t) \sim \sum_{m=0}^{\infty} c_m \frac{t^m}{\Gamma(\frac{m+1}{q})}$ as $t \to 0$ in $S(\theta,\alpha)$.

Now let f and g be holomorphic functions in a neighbourhood of ∞ in $S(-\theta, \alpha + \frac{\pi}{q})$ and assume that both are $O(x^{-\varepsilon})$ as $x \to \infty$ in $S(-\theta, \alpha + \frac{\pi}{q})$ for some $\varepsilon > 0$. If we denote $F = \mathcal{B}_q f$ and $G = \mathcal{B}_q g$, then one can prove that

$$\mathcal{B}_q(f \cdot g) = F *_q G, \tag{1.4.3}$$

where the so-called *q-convolution* $F *_q G$ is defined by

$$(F *_q G)(t) := \int_0^t F\big((t^q - \sigma^q)^{1/q}\big) G(\sigma)\, \mathrm{d}(\sigma^q).$$

A trivial, but useful example of q-convolution is the following:

$$t^{r-q} *_q t^{s-q} = B\Big(\frac{r}{q}, \frac{s}{q}\Big) t^{r+s-q}, \quad \Re r, \Re s > 0, \tag{1.4.4}$$

where B is the beta function.

1.4.2 The Laplace Transform of Order q

Let F be a holomorphic function which is of exponential growth of order $\leqslant q$ in $S(\theta,\alpha)$. Moreover, assume that $F(t) = O(t^{\varepsilon - q})$ as $t \to 0$ in $S(\theta,\alpha)$ for some $\varepsilon > 0$. Then the Laplace transform of order q of F is defined by

$$(\mathcal{L}_q F)(x) := \int_0^{\infty e^{i\psi}} F(t)\, e^{-(xt)^q} \mathrm{d}(t^q), \tag{1.4.5}$$

for arbitrary $\psi \in (\theta - \alpha/2, \theta + \alpha/2)$. Obviously this integral converges for $\Re(x^q e^{iq\psi})$ sufficiently large, so a priori the integral represents a holomorphic function in a neighbourhood of ∞ in $S(-\psi, \pi/q)$. By varying ψ in the interval above, we end up with a Laplace transform $\mathcal{L}_q F$ that is holomorphic in a neighbourhood of ∞ in $S(-\theta, \alpha + \frac{\pi}{q})$.

Given a convergent or divergent series $\hat{F}(t) = \sum_{m=0}^{\infty} c_m t^{m+1-q}$, the formal Laplace transform of order q of $\hat{F}$ is defined by

$$(\hat{\mathcal{L}}_q \hat{F})(x) := \sum_{m=0}^{\infty} c_m \, \Gamma\Big(\frac{m+1}{q}\Big) \, x^{-m-1}. \tag{1.4.6}$$

Again, if $F(t) \sim \hat{F}(t)$ as $t \to 0$ on $S(\theta, \alpha)$ and the above assumptions on F are satisfied, then $(\mathcal{L}_q F)(x) \sim (\hat{\mathcal{L}}_q \hat{F})(x)$ as $x \to \infty$ in $S(-\theta, \alpha + \frac{\pi}{q})$ and in that case $\mathcal{B}_q \mathcal{L}_q F = F$. If, on the other hand, f is a holomorphic function in a neighbourhood of ∞ in the sector $S(-\theta, \alpha + \frac{\pi}{q})$ and $f(x) \sim \sum_{m=0}^{\infty} c_m x^{-m-1}$ as $x \to \infty$ in this sector, then $\mathcal{L}_q \mathcal{B}_q f = f$.

If F and G are holomorphic and of exponential growth of order $\leqslant q$ in $S(\theta, \alpha)$ and $F(t)$, $G(t) = O(t^{\varepsilon - q})$ as $t \to 0$ on $S(\theta, \alpha)$ for some $\varepsilon > 0$, then the q-convolution of F and G is defined and

$$\mathcal{L}_q(F *_q G) = \mathcal{L}_q F \cdot \mathcal{L}_q G \tag{1.4.7}$$

in a neighbourhood of ∞ in $S(-\theta, \alpha + \pi/q)$.

1.4.3 Definition of Multisummability

In order to give a proper definition of multisummability we first have to introduce another integral operator: Écalle's *acceleration operator*. For $q' > q > 0$ this acceleration operator $\mathcal{A}_{q',q}$ is defined by

$$\mathcal{A}_{q',q} := \mathcal{B}_{q'} \mathcal{L}_q.$$

This operator makes sense on the space of functions $F : S(\theta, \alpha) \to \mathbb{C}$ which are of exponential growth of order $\leqslant q$ in $S(\theta, \alpha)$ and which satisfy $F(t) = O(t^{\varepsilon - q})$ as $t \to 0$ in $S(\theta, \alpha)$ for some $\varepsilon > 0$. Écalle has shown that this operator can be extended to functions with the same conditions except that 'exponential growth of order $\leqslant q$' is replaced by 'exponential growth of order $\leqslant \kappa$', where $\frac{1}{\kappa} := \frac{1}{q} - \frac{1}{q'}$ i.e. $\kappa = \frac{q'q}{q'-q}$. For such a function F, the accelerate $\mathcal{A}_{q',q} F$ is holomorphic in a neighbourhood of 0 in $S(\theta, \alpha + \pi/\kappa)$ and can be represented by

$$(\mathcal{A}_{q',q} F)(t) = t^{-q'} \int_0^{\infty e^{i\psi}} C_{q'/q}\big((\xi/t)^q\big) \, F(\xi) \, \mathrm{d}(\xi^q), \tag{1.4.8}$$

where $\psi \in (\theta - \alpha/2, \theta + \alpha/2)$. Here $C_{q'/q}$ is a special case of Écalle's function C_α, defined for $\alpha > 1$ by

$$C_\alpha(t) := \sum_{m=1}^{\infty} \frac{(-t)^m}{m! \, \Gamma(-m/\alpha)}, \quad t \in \mathbb{C}. \tag{1.4.9}$$

If $F(t) \sim \hat{F}(t) := \sum_{m=0}^{\infty} c_m t^{m+1-q}$ as $t \to 0$ on $S(\theta, \alpha)$, then

$$(\mathcal{A}_{q',q} F)(t) \sim \sum_{m=0}^{\infty} c_m \frac{\Gamma(\frac{m+1}{q})}{\Gamma(\frac{m+1}{q'})} t^{m+1-q'} =: (\hat{\mathcal{A}}_{q',q} \hat{F})(t),$$

as $t \to 0$ in $S(\theta, \alpha + \pi/\kappa)$. Moreover, if F and G are functions for which the accelerate is defined, then one can prove that

$$\mathcal{A}_{q',q}(F *_q G) = (\mathcal{A}_{q',q}F) *_{q'} (\mathcal{A}_{q',q}G).$$

Definition 1.4.1 (Multisummability) *Let $\hat{f} \in x^{-1}\mathbb{C}[[x^{-1}]]$ and let r be a natural number. Let $\mathbf{q} = (q_1, q_2, \ldots, q_r)$, $0 < q_r < q_{r-1} < \cdots < q_1$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_r) \in \mathbb{R}^r$ and let $S_j = S(-\theta_j, \alpha_j + \pi/q_j)$, $\alpha_j > 0$, $j = 1, 2, \ldots, r$. Moreover, define*

$$q_0 := \infty, \quad \kappa_j = \left(q_j^{-1} - q_{j-1}^{-1}\right)^{-1}, \quad j = 1, 2, \ldots, r.$$

Then $\hat{f}$ is said to be $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta}$ (or $\mathbf{q}$-summable on the multi-sector $\mathbf{S} := (S_1, S_2, \ldots, S_r)$) if

(i) *For all $j \in \{2, 3, \ldots, r\}$: $S_{j-1} \subset S_j$.*

(ii) *The series $\hat{\mathcal{B}}_{q_r}\hat{f}$ is convergent in $\Delta(0, \rho)\backslash\{0\}$ for some $\rho > 0$. Denote g_r the sum of this convergent series.*

(iii) *For $j = r, r-1, \ldots, 1$ respectively, the function g_j can be analytically continued and is of exponential growth of order $\leqslant \kappa_j$ in $S(\theta_j, \alpha_j)$. Here g_{j-1}, $j = r, r-1, \ldots, 2$, is defined in a neighbourhood of 0 in $S(\theta_j, \alpha_j + \pi/\kappa_j)$ by $g_{j-1} := \mathcal{A}_{q_{j-1}, q_j} g_j$.*

If these conditions are fulfilled, then the $\mathbf{q}$-sum (or multi-sum) of the formal series $\hat{f}$ is defined by $S_{\mathbf{q},\boldsymbol{\theta}}\hat{f} := \mathcal{L}_{q_1} g_1$.

This sum is holomorphic in a neighbourhood U of ∞ in S_1 and it satisfies $(S_{\mathbf{q},\boldsymbol{\theta}}\hat{f})(x) \sim \hat{f}(x)$ as $x \to \infty$ in U. The summation operator $S_{\mathbf{q},\boldsymbol{\theta}}$ is injective. Moreover, Borel summability coincides with 1-summability, since $\mathcal{B} = \mathcal{B}_1$ and $\mathcal{L} = \mathcal{L}_1$. If $\hat{f} \in x^{-1}\mathbb{C}[[x^{-1}]]$ is $\mathbf{q}$-summable in all but countably many multi-directions, we just call $\hat{f}$ $\mathbf{q}$-summable.

As in the case of Borel summability, the set of multisummable power series is closed under addition, multiplication and differentiation and every series that converges in a neighbourhood of ∞ is $\mathbf{q}$-summable in every multi-direction with multi-sum equal to the sum of the convergent series. A series $\hat{f}(x) := \sum_{m=0}^{\infty} c_m x^{-m}$ is $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta}$ if $\hat{g}(x) := \sum_{m=1}^{\infty} c_m x^{-m}$ is $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta}$ and it that case $S_{\mathbf{q},\boldsymbol{\theta}}\hat{f} = c_0 + S_{\mathbf{q},\boldsymbol{\theta}}\hat{g}$.

A proof of the following lemma can be found in [Bal94]. Another proof, that uses cohomology arguments, is given by Malgrange and Ramis in [MalR92].

Lemma 1.4.2 *For natural n let $h \in \mathbb{C}\{x^{-1}, y_1, \ldots, y_n\}$. Choosing arbitrary formal power series $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n$ in x^{-1} with vanishing constant terms, one can (formally) define a power series*

$$\hat{g}(x) := h\big(x, \hat{f}_1(x), \ldots, \hat{f}_n(x)\big).$$

Let $\mathbf{q} := (q_1, q_2, \ldots, q_r)$ with $1/2 < q_r < q_{r-1} < \cdots < q_1$ and assume that the formal series $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_r$ are $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta} \in \mathbb{R}^r$. Then $\hat{g}$ is $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta}$.

1.5 Two Illustrative Examples

In this section we illustrate the occurrence of transseries and singularities at Stokes lines of small solutions in examples of a Riccati differential equation and a Riccati difference equation. However, these equations may also be discussed without the use of transseries.

Example 1.5.1 In [CK02] Costin and Kruskal considered the Bernoulli differential equation $y' - y + y^2 = 0$. Following a suggestion of M. van der Put we consider here the differential equation

$$y'(x) + y(x) + y^2(x) + x^{-2} = 0 \tag{1.5.1}$$

for x in a neighbourhood of ∞ and we will restrict ourselves to complex valued solutions y which tend to 0 as $x \to \infty$.

If we write $y = \frac{f'}{f}$ or $f' = yf$ for some function f, then $f'' = y'f + y^2 f$. Using (1.5.1) we see that f satisfies the equation

$$f''(x) + f'(x) + x^{-2} f(x) = 0. \tag{1.5.2}$$

In general (1.5.1) is called the *Riccati equation* of (1.5.2). From [Olv74], sections 1 and 2 (in particular theorems 2.1 and 2.2) in chapter 7, we conclude that there exist formal series $\hat{f}_0, \hat{f}_1 \in \mathbb{C}[[x^{-1}]]$ and holomorphic solutions f_0 and g_0 with $f_0(x) \sim \hat{f}_0(x)$ as $x \to \infty$ in $S(\pi, 3\pi)$ and $e^x g_0(x) \sim \hat{f}_1(x)$ as $x \to \infty$ in $S(0, 3\pi)$. Moreover, the constant terms in $\hat{f}_0$ and $\hat{f}_1$ may be chosen arbitrary, and we will give them the value 1. With this prescription both the formal solutions $\hat{f}_0$ and $\hat{f}_1$ and the holomorphic solutions f_0 and g_0 turn out to be unique. Now observe that if $e^{-x}\hat{f}_1(x)$ satisfies (1.5.2), then $\hat{f}_1$ satisfies $f_1''(x) - f_1'(x) + x^{-2} f_1(x) = 0$, so $\hat{f}_1(x) = \hat{f}_0(-x)$.

Substitution of $\hat{f}_0(x) = \sum_{m=0}^{\infty} c_m x^{-m}$ in (1.5.2) gives

$$\sum_{m=3}^{\infty} (m-1)(m-2) c_{m-2}\, x^{-m} - \sum_{m=2}^{\infty} (m-1) c_{m-1}\, x^{-m} + \sum_{m=2}^{\infty} c_{m-2}\, x^{-m} = 0$$

and comparing coefficients of x^{-m}, $m \geqslant 0$, leads to the recurrence relation

$$c_m = \frac{m^2 - m + 1}{m} c_{m-1}, \quad m \geqslant 1.$$

Given the prescription $c_0 = 1$, the other coefficients are determined uniquely. In fact, if $\zeta = e^{\pm\pi i/3}$, then $c_m = \frac{(m-\zeta)(m-1+\zeta)}{m} c_{m-1}$ and thus $c_m = \frac{(\zeta)_m (1-\zeta)_m}{m!}$ [4]. So (1.5.2) possesses the following two formal solutions:

$$\hat{f}_0(x) = \sum_{m=0}^{\infty} \frac{(\zeta)_m (1-\zeta)_m}{m!} x^{-m} \quad \text{and} \quad \hat{g}_0(x) = e^{-x} \sum_{m=0}^{\infty} (-1)^m \frac{(\zeta)_m (1-\zeta)_m}{m!} x^{-m}.$$

Hence, one may conclude that $\alpha \hat{f}_0(x) - \beta e^{-x} \hat{f}_1(x)$, with α and β arbitrary complex constants, again is a formal solution of (1.5.2). If we define z to be the general holomorphic

[4] Here we used *Pochhammer's symbol*: $(a)_0 = 1$, $(a)_m = a(a+1)\cdots(a+m-1)$, $m \in \mathbb{N}_+$.

solution of $z'(x) + z(x) = 0$, i.e. $z(x) = \beta e^{-x}$, $\beta \in \mathbb{C}$, then the formal solution of (1.5.2) described above can be written as $\alpha \hat{f}_0(x) - z(x)\hat{f}_1(x)$. Now if we choose $\alpha = 0$, then the corresponding formal solution of (1.5.1) equals $\hat{y}(x) = \frac{(z(x)\hat{f}_1(x))'}{z(x)\hat{f}_1(x)} = \frac{\hat{f}_1'(x)}{\hat{f}_1(x)} - 1$. In that case $\hat{y}(\infty) \neq 0$, which means that if $y(x)$ is a solution of (1.5.1) that asymptotically equals $\hat{y}(x)$ as $x \to \infty$, then $y(x) \not\to 0$ as $x \to \infty$. Since we only consider solutions y that tend to 0 as $x \to \infty$, we thus have $\alpha \neq 0$ and by taking $\alpha = 1$ we get

$$\hat{y}(x) = \frac{\hat{f}_0'(x) - z(x)\big(\hat{f}_1'(x) - \hat{f}_1(x)\big)}{\hat{f}_0(x) - z(x)\hat{f}_1(x)} \tag{1.5.3}$$

as formal solution of (1.5.1). By construction, the *formal* transformation

$$y = \hat{P}(x,z) := \frac{\hat{f}_0'(x) - z\big(\hat{f}_1'(x) - \hat{f}_1(x)\big)}{\hat{f}_0(x) - z\hat{f}_1(x)}$$

transforms (1.5.1) into the *normal form* $z'(x) + z(x) = 0$. Now, the expression $\hat{P}(x,z)$ can formally be rewritten as a power series in z,

$$\hat{P}(x,z) = \sum_{k=0}^{\infty} \hat{y}_k(x)\, z^k,$$

with $\hat{y}_0(x) = \frac{\hat{f}_0'(x)}{\hat{f}_0(x)}$ and $\hat{y}_k(x) = \left(\frac{\hat{f}_1(x)}{\hat{f}_0(x)}\right)^k \left(\frac{\hat{f}_0'(x)}{\hat{f}_0(x)} - \frac{\hat{f}_1'(x)}{\hat{f}_1(x)} + 1\right)$ for $k \geqslant 1$. When we substitute the solution $z(x) = \beta e^{-x}$ of the normal form $z'(x) + z(x) = 0$ into the expansion of $\hat{P}(x,z)$, we obtain an example of a *transseries*, i.e. a formal exponential series solution of (1.5.1):

$$\hat{y}(x) = \sum_{k=0}^{\infty} \beta^k e^{-kx} \hat{y}_k(x). \tag{1.5.4}$$

As pointed out in the beginning of this example we can conclude from [Olv74] that the formal transformation $y = \hat{P}(x,z)$ can be lifted to an actual transformation, because there exist lifts $f_0(x)$ of $\hat{f}_0(x)$ and $f_1(x) := e^x g_0(x)$ of $\hat{f}_1(x)$. So, the *transformation*

$$y = P(x,z), \quad \text{with } P(x,z) := \frac{f_0'(x) - z\big(f_1'(x) - f_1(x)\big)}{f_0(x) - zf_1(x)},$$

is well defined for x in a neighbourhood of ∞ in $S := S(\pi/2, 2\pi)$ and z in a neighbourhood of 0, and transforms the differential equation (1.5.1) in a neighbourhood of ∞ in S into the normal form $z'(x) + z(x) = 0$. Moreover, if we define

$$y(x) := \frac{f_0'(x) - \beta e^{-x}(f_1'(x) - f_1(x))}{f_0(x) - \beta e^{-x} f_1(x)}, \tag{1.5.5}$$

then, for each $\beta \in \mathbb{C}$, y is an actual *solution* of (1.5.1), which is holomorphic in a neighbourhood of ∞ in the region $R := S \backslash \{x \in \mathbb{C}^* \mid f_0(x) = \beta e^{-x} f_1(x)\}$.

We can prove that both $\hat{f}_0$ and $\hat{f}_1$ are Borel summable: the formal Borel transform of $x^{-\zeta}\hat{f}_0(x)$, with $\zeta = e^{\pm\pi i/3}$, equals

$$\hat{\mathcal{B}}[x^{-\zeta}\hat{f}_0](t) = \sum_{m=0}^{\infty} \frac{c_m}{\Gamma(m+\zeta)}\, t^{m+\zeta-1} = \sum_{m=0}^{\infty} \frac{(1-\zeta)_m}{m!\Gamma(\zeta)}\, t^{m+\zeta-1} = \frac{\{t(1-t)\}^{\zeta-1}}{\Gamma(\zeta)},$$

where the last equality follows from the fact that $\frac{(1-\zeta)_m}{m!} = (-1)^m\binom{\zeta-1}{m}$. The function $t \mapsto \{t(1-t)\}^{\zeta-1}\Gamma^{-1}(\zeta)$ is holomorphic for $t \neq 0, 1$, integrable at $t = 0$ and of at most exponential growth in every sector not containing the positive real axis. So $x^{-\zeta}\hat{f}_0(x)$, and thus $\hat{f}_0$, is Borel summable and the Borel sum of $\hat{f}_0$ is holomorphic in a neighbourhood of ∞ in $S(\pi, 3\pi)$. Since f_0 is the unique solution of (1.5.2) with $f_0(x) \sim \hat{f}_0(x)$ as $x \to \infty$ in $S(\pi, 3\pi)$, the Borel sum of $\hat{f}_0$ coincides with f_0, so

$$f_0(x) = \frac{x^\zeta}{\Gamma(\zeta)} \int_0^{\infty e^{i\theta}} e^{-xt}\{t(1-t)\}^{\zeta-1}\mathrm{d}t, \quad |\arg x + \theta| < \frac{\pi}{2}, \quad -2\pi < \theta < 0. \tag{1.5.6}$$

Obviously, $f_0(xe^{\pi i})$ is the Borel sum of $\hat{f}_1(x)$ on $S(0, 3\pi)$ and coincides with $f_1(x) = e^x g_0(x)$. Hence, $e^{-x}f_0(xe^{\pi i})$ is a solution of (1.5.2).

Another way to reach the last conclusion is by observing that in (1.5.6) it is allowed to take $\theta = 0$ and $\theta = -2\pi$, provided that $\arg(1-t) = \pi$, respectively $-\pi$, for $t > 1$. In particular for $x > 0$, $\arg x = 0$, we have

$$f_0(x) = \frac{x^\zeta}{\Gamma(\zeta)}\left\{\int_0^1 e^{-xt}\{t(1-t)\}^{\zeta-1}\mathrm{d}t + \int_1^\infty e^{-xt}\{te^{\pi i}(t-1)\}^{\zeta-1}\mathrm{d}t\right\}$$

and

$$f_0(xe^{2\pi i}) = \frac{(xe^{2\pi i})^\zeta}{\Gamma(\zeta)}\left\{\int_0^1 e^{-xt}\{e^{-2\pi i}t(1-t)\}^{\zeta-1}\mathrm{d}t + \int_1^\infty e^{-xt}\{e^{-2\pi i}te^{-\pi i}(t-1)\}^{\zeta-1}\mathrm{d}t\right\}.$$

Since the two integrals from $t = 0$ to $t = 1$ coincide we get

$$\begin{aligned} f_0(x) - f_0(xe^{2\pi i}) &= \frac{x^\zeta}{\Gamma(\zeta)}\{e^{\pi i(\zeta-1)} - e^{-\pi i(\zeta-1)}\}\int_1^\infty e^{-xt}\{t(t-1)\}^{\zeta-1}\mathrm{d}t \\ &= 2i\sin(\pi\zeta)\frac{x^\zeta}{\Gamma(\zeta)}e^{-x}\int_0^{\infty e^{-\pi i}} e^{xs}\{(1-s)e^{\pi i}s\}^{\zeta-1}\mathrm{d}s \\ &= -2i\sin(\pi\zeta)e^{-x}f_0(xe^{\pi i}), \end{aligned}$$

where the substitution $t = 1 + e^{\pi i}s$ is used to obtain the second equality. By analytic continuation we then deduce that

$$f_0(x) - f_0(xe^{2\pi i}) = -2i\sin(\pi\zeta)e^{-x}f_0(xe^{\pi i}), \quad |\arg x| < \frac{\pi}{2}. \tag{1.5.7}$$

Since the left-hand side of the preceding formula is a solution of (1.5.2), the expression $e^{-x}f_0(xe^{\pi i})$ also is a solution of this equation. The formula (1.5.7) is a connection formula for f_0.

There also is a remarkable correspondence with modified Bessel functions, as follows: the integral representation of the modified Bessel function of the third kind[5], K_ν, is given by (cf. [Wat44], p. 206)

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\frac{e^{-x}}{\Gamma(\nu+1/2)}\int_0^\infty e^{-u}u^{\nu-1/2}\left(1+\frac{u}{2x}\right)^{\nu-1/2}\mathrm{d}u$$

[5]In some textbooks this function is also called *Macdonald's function*.

and if we choose $\nu = \zeta - 1/2$ and put $u = e^{\pi i} tx$, then one easily infers the following correspondence with f_0

$$\begin{cases} e^{-x} f_0(xe^{\pi i}) = \sqrt{\frac{x}{\pi}}\, e^{-x/2} K_{\zeta-1/2}(x/2) \\ f_0(x) = \sqrt{\frac{xe^{-\pi i}}{\pi}}\, e^{-x/2} K_{\zeta-1/2}(xe^{-\pi i}/2). \end{cases} \tag{1.5.8}$$

So the asymptotic expansion of f_0 also gives an asymptotic expansion for K_ν (which coincides with the one Watson gives in [Wat44], p. 207) and the connection formula (1.5.7) gives a connection formula for K_ν.

Finally, we mention that the differential equation (1.5.2) can be reduced to a modified Bessel equation of order $\nu = \zeta - 1/2$, by means of the substitution $f(x) = \sqrt{x} e^{-x/2} u(\pm ix/2)$. Then u satisfies a modified Bessel equation with as general solution the so-called cylinder functions $C_{\zeta-1/2}$ (compare [Wat44], p. 98, formulae (15) and (16) with $\varphi(x) = e^{-x}$ and $\psi(x) = \pm\frac{ix}{2}$). For these cylinder functions one can choose the functions $K_{\zeta-1/2}(e^{\pm\pi i/2} t)$ and with this particular choice we get, via (1.5.8), $f_0(x)$ and $e^{-x} f_0(xe^{\pi i})$, except for a constant factor, as solutions for (1.5.2).

As both $\hat{f}_0$ and $\hat{f}_1$ are Borel summable, the same holds for each $\hat{y}_k$ and their Borel sums

$$y_0(x) = \frac{f_0'(x)}{f_0(x)} \quad \text{and} \quad y_k(x) = \left(\frac{f_1(x)}{f_0(x)}\right)^k \left(\frac{f_0'(x)}{f_0(x)} - \frac{f_1'(x)}{f_1(x)} + 1\right), \quad k \geqslant 1,$$

are holomorphic in a neighbourhood of ∞ in S, but of course this does not guarantee convergence of the series $\sum_{k=0}^{\infty} \beta^k e^{-kx} y_k(x)$. In fact, one can prove that the formal series $\hat{P}(x,z)$ is Borel summable with respect to x, *provided that* $|z|$ *is small enough*, and then its sum equals $\sum_{k=0}^{\infty} y_k(x) z^k$. This statement is shown in a more general context by Braaksma in [Bra01]. Hence, we only are allowed to write y, as given in (1.5.5), in the form of a convergent transseries $\sum_{k=0}^{\infty} \beta^k e^{-kx} y_k(x)$ if βe^{-x} decreases as $x \to \infty$, which is the case on $R \cap \{x \in \mathbb{C}^* \mid \Re x > 0\}$.

From (1.5.5) we see that y might be singular at those points $x \in \mathbb{C}$ satisfying the equation $f_0(x) = \beta e^{-x} f_1(x)$. Then $e^x = \beta f_0(-x) f_0^{-1}(x)$ and thus

$$x = \ln\beta + \ln\bigl(f_0(-x) f_0^{-1}(x)\bigr) + 2n\pi i, \quad n \in \mathbb{Z}. \tag{1.5.9}$$

This equation can be rewritten as

$$\bigl[x - \ln\bigl(f_0(-x) f_0^{-1}(x)\bigr)\bigr]^{-1} = (\ln\beta + 2n\pi i)^{-1} =: \gamma_n, \quad n \in \mathbb{Z}.$$

Putting $x = t^{-1}$ we infer that the latter equation is equivalent to

$$t = \gamma_n\bigl(1 + \psi(t)\bigr) := h(t), \quad \text{with} \quad \psi(t) = -t \ln\Bigl(\frac{f_0(-t^{-1})}{f_0(t^{-1})}\Bigr). \tag{1.5.10}$$

One should notice that ψ is holomorphic for t in a neighbourhood of 0 in $\tilde{S} := S(-\pi/2, 2\pi)$. Hence, ψ certainly is holomorphic $G := \{t \in \mathbb{C}^* \mid 0 < |t| < r,\ -3\pi/2 + \varepsilon < \arg t < \pi/2 - \varepsilon\}$ for some $r > 0$ and $\varepsilon > 0$. From the asymptotic behaviour of f_0 and f_1 we easily deduce that $\psi(t) = o(t)$ as $t \to 0$ in $\tilde{S}$, so $\psi'(t) = o(1)$ as $t \to 0$ in $\tilde{S}$, and $K := \sup_{t \in G} |\psi'(t)| < \infty$. From this we easily infer that $t \mapsto h(t)$ is a contraction on G provided that γ_n belongs to some subset G_1 of G, $|\gamma_n| < 1/K$ and r sufficiently small (i.e. so small that $h(t) \in G$ for $t \in G$). The requirement $\gamma_n \in G_1$, $|\gamma_n| < 1/K$, can only be fulfilled for $n \in \mathbb{N}$, n large enough. Hence, (1.5.10) has a unique solution $t = t_n$ for every $n \in \mathbb{N}$ large enough. So, if we define $x_n := t_n^{-1}$, then $x = x_n$ is a solution of (1.5.9), provided that $n \in \mathbb{N}$ is large enough.

As $\ln\big(f_0(-x)f_0^{-1}(x)\big) = O(x^{-1})$, $x \to \infty$ in S, we deduce that $x_n = \ln\beta + 2n\pi i + O(x_n^{-1})$ and using this we see that the solution y of (1.5.1) might be singular at points $x = x_n$ with

$$x_n = \ln\beta + 2n\pi i + O(n^{-1}), \quad n \in \mathbb{N},\ n \to \infty. \tag{1.5.11}$$

Thus we conclude that if $\beta \neq 0$, the solution y might be singular at a distance at most $O(n^{-1})$ of $\ln\beta + 2n\pi i$, as $n \in \mathbb{N}$, $n \to \infty$. Here we recognise a more general result of O. Costin and R.D. Costin on the formation of singularities of solutions of nonlinear differential equations in so-called *Stokes directions*[6]. For more details on this result the reader is referred to [CC01], theorem 2.

Example 1.5.2 As an analogue of the preceding example we consider the difference equation

$$y(x+1) + y(x)y(x+1) = e^{-1}y(x) + x^{-2} \tag{1.5.12}$$

for x in a neighbourhood of ∞. Again we will only consider complex valued solutions y such that $y(x) \to 0$ as $x \to \infty$. Notice that the linearised equation $z(x+1) = e^{-1}z(x)$ now has as general holomorphic solution $z(x) = \beta(x)e^{-x}$ with β an arbitrary 1-periodic holomorphic function, which is the analogue of the general solution of the linearised differential equation in the preceding example. This is the reason to put e^{-1} in front of $y(x)$ in the right-hand side of (1.5.12).

If we write $y(x) = \frac{f(x+1)}{f(x)} - 1$ for some function f (compare [MT51], p. 346), then a straightforward calculation shows that f satisfies the following second order linear difference equation

$$f(x+2) = (e^{-1}+1)f(x+1) + (x^{-2} - e^{-1})f(x). \tag{1.5.13}$$

As in the preceding example we first try to find formal solutions $\hat{f}_0(x) \in \mathbb{C}[[x^{-1}]]$ and $\hat{g}_0(x) = e^{-x}\hat{f}_1(x)$, with $\hat{f}_1 \in \mathbb{C}[[x^{-1}]]$.

For $a \in \mathbb{C}$ and $m \in \mathbb{N}_+$ we have $(a+x)^{-m} = x^{-m}(1+ax^{-1})^{-m} = \sum_{k=0}^{\infty}\binom{-m}{k}a^k x^{-m-k}$. So substituting the series $\sum_{m=0}^{\infty} c_m x^{-m}$ in (1.5.13) and comparing coefficients of x^{-m}, $m \geqslant 0$, we find the relations

$$\left\{\begin{array}{rcl} c_0 &=& (e^{-1}+1)c_0 - e^{-1}c_0 \\ c_1 &=& (e^{-1}+1)c_1 - e^{-1}c_1 \\ \sum_{k=1}^{m}\binom{-k}{m-k}2^{m-k}c_k &=& \sum_{k=1}^{m}\binom{-k}{m-k}(1+e^{-1})c_k - e^{-1}c_m + c_{m-2}, \quad m \geqslant 2. \end{array}\right.$$

[6]If σ is a singular direction (in our case $\sigma = 0$), then $-\sigma \pm \pi/2$ is called a *Stokes direction* (in our case the imaginary axis). In [CC01] however, Costin and Costin use the term *anti-Stokes direction* for such a half line.

The first two equations are trivially satisfied and the third equation is equivalent to

$$\sum_{k=1}^{m-1}\binom{-k}{m-k}2^{m-k}c_k=\sum_{k=1}^{m-1}\binom{-k}{m-k}(1+e^{-1})c_k+c_{m-2}, \tag{1.5.14}$$

which for $m=2$ reduces to $-2c_1=-(1+e^{-1})c_1+c_0$. As in the preceding example we prescribe $c_0:=1$. Then c_1 is determined uniquely. Now assume that c_m has been found for $m\in\{0,1,\dots,\ell-1\}$ for some $\ell\geqslant 2$, then c_ℓ can be determined uniquely from (1.5.14) with $m=\ell+1$.

To find the other formal solution we first observe that $\hat{f}_1$ satisfies

$$f_1(x+2)=(1+e)f_1(x+1)+\big(e^2x^{-2}-e\big)f_1(x) \tag{1.5.15}$$

and in a similar way as before we can construct a unique formal solution $\hat{f}_1\in\mathbb{C}^n[[x^{-1}]]$ of (1.5.15), if we prescribe its constant term to be equal to 1. Hence, $\alpha(x)\hat{f}_0(x)-\beta(x)e^{-x}\hat{f}_1(x)$, with α and β holomorphic 1-periodic functions, also is a formal solution of (1.5.13). In the following we assume that α is invertible. Then without loss of generality we can assume that $\alpha\equiv 1$ and

$$\hat{y}(x)=\frac{\hat{f}_0(x+1)-e^{-1}z(x)\hat{f}_1(x+1)}{\hat{f}_0(x)-z(x)\hat{f}_1(x)}-1$$

is a formal solution of (1.5.12). By construction the formal transformation

$$y=\hat{P}(x,z):=\frac{\hat{f}_0(x+1)-e^{-1}z\hat{f}_1(x+1)}{\hat{f}_0(x)-z\hat{f}_1(x)}-1$$

transforms the difference equation (1.5.12) into the normal form $z(x+1)=e^{-1}z(x)$ and, as in the preceding example, the expression $\hat{P}(x,z)$ can formally be expanded as

$$\hat{P}(x,z)=\sum_{k=0}^{\infty}\hat{y}_k(x)\,z^k,$$

with $\hat{y}_0(x)=\frac{\hat{f}_0(x+1)}{\hat{f}_0(x)}-1$ and $\hat{y}_k(x)=\Big(\frac{\hat{f}_1(x)}{\hat{f}_0(x)}\Big)^k\Big(\frac{\hat{f}_0(x+1)}{\hat{f}_0(x)}-e^{-1}\frac{\hat{f}_1(x+1)}{\hat{f}_1(x)}\Big)$ for $k\geqslant 1$. When we substitute the solution $z(x)=\beta(x)e^{-x}$ of the normal form $z(x+1)=e^{-1}z(x)$ into the expansion of $\hat{P}(x,z)$, we obtain a transseries solution of (1.5.12):

$$\hat{y}(x)=\sum_{k=0}^{\infty}\beta^k(x)e^{-kx}\hat{y}_k(x). \tag{1.5.16}$$

To show the Borel summability of for example $\hat{f}_0$, we put $F(x):=\big(f_0(x),f_0(x+1)\big)^t$ and we see that F satisfies

$$F(x+1)=A(x)F(x),\qquad \text{with } A(x)=\begin{pmatrix}0 & 1\\ x^{-2}-e^{-1} & e^{-1}+1\end{pmatrix}.$$

The constant matrix $A(\infty)$ has two different eigenvalues, namely $\lambda = e^{-1}$ and $\lambda = 1$. If Q is a matrix with the property $QA(\infty)Q^{-1} = \mathrm{diag}\{1, e^{-1}\}$, then $\tilde{F} := QF$ satisfies $\tilde{F}(x+1) = QA(x)Q^{-1}\tilde{F}(x)$, which is in the form that Braaksma considered in [Bra80]. Using theorem 2 and 3 from this article one may show that $\hat{f}_0$ is Borel summable in every direction $-\theta$, with $\theta \neq \pm\frac{\pi}{2}, \arg(1+2l\pi i)$, $l \in \mathbb{Z}$ (cf. also [Hor16, Hor18]). In a similar way one might observe that $\hat{f}_1$ is Borel summable in every direction $-\theta$, with $\theta \neq \pm\frac{\pi}{2}, \arg(-1+2l\pi i)$, $l \in \mathbb{Z}$. Note that if f_0 and f_1 are the Borel sums of $\hat{f}_0$ and $\hat{f}_1$ respectively, then in particular f_0 exists and is holomorphic in a neighbourhood of ∞ in $S(\pi, 2\pi)$ and f_1 exists and is holomorphic in a neighbourhood of ∞ in $S(0, 2\pi)$.

In the following S denotes the sector $S(\pi/2, \pi)$. Then both f_0 and f_1 are holomorphic in a neighbourhood of ∞ in S and $f_0(x) \sim \hat{f}_0(x)$ and $f_1(x) \sim \hat{f}_1(x)$ as $x \to \infty$ in S. Hence, the *transformation*

$$y = P(x, z) := \frac{f_0(x+1) - e^{-1} z f_1(x+1)}{f_0(x) - z f_1(x)} - 1$$

is well defined for x in a neighbourhood of ∞ in S and z in a neighbourhood of 0 and transforms the difference equation (1.5.12) into the normal form $z(x+1) = e^{-1}z(x)$. Moreover, if we define

$$y(x) := \frac{f_0(x+1) - e^{-1}\beta(x)e^{-x} f_1(x+1)}{f_0(x) - \beta(x)e^{-x} f_1(x)} - 1, \tag{1.5.17}$$

then y is an actual *solution* of (1.5.12), which is holomorphic in a neighbourhood of ∞ in the region $R := S \backslash \{x \in \mathbb{C}^* \mid f_0(x) = \beta(x)e^{-x} f_1(x)\}$.

As both $\hat{f}_0$ and $\hat{f}_1$ are Borel summable, the same holds for each $\hat{y}_k$ and their Borel sums

$$y_0(x) = \frac{f_0(x+1)}{f_0(x)} - 1 \quad \text{and} \quad y_k(x) = \left(\frac{f_1(x)}{f_0(x)}\right)^k \left(\frac{f_0(x+1)}{f_0(x)} - e^{-1}\frac{f_1(x+1)}{f_1(x)}\right), \quad k \geqslant 1,$$

are holomorphic in a neighbourhood of ∞ in S. However, as in the preceding example, this does not guarantee convergence of the corresponding transseries. Again one can prove that the formal series $\hat{P}(x, z)$ is Borel summable with respect to x, *provided that* $|z|$ *is small enough*, and then its sum equals $\sum_{k=0}^{\infty} y_k(x) z^k$, which is shown by Braaksma in [Bra01]. Hence, we only are allowed to write the solution y, as given in (1.5.17), in the form of a convergent transseries, if $\beta(x)e^{-x}$ decreases as $x \to \infty$.

Now let us assume that $\beta(x)e^{-x}$ indeed decreases to 0 as $x \to \infty$ in S, and thus in every sub-sector S_1 of S, and assume that the Fourier expansion of β is given by $\sum_{h=-\infty}^{\infty} \beta_h e^{2\pi i h x}$. Then

$$\beta_h = \int_0^1 \beta(x+t) e^{-(x+t)} e^{(1-2\pi i h)(x+t)} \mathrm{d}t,$$

if $[x, x+1] \in S_1$. Let us write $S_1 = \{x \in \mathbb{C}^* \mid \varphi_- < \arg x < \varphi_+\}$, with $0 < \varphi_- < \varphi_+ < \pi$. The integral above tends to 0 as $x \to \infty$, $\arg x = \varphi$, if $\cos\varphi + 2\pi h \sin\varphi < 0$. Now, let h_- be equal to $\frac{-1}{2\pi}\cot\varphi_-$ if $\frac{-1}{2\pi}\cot\varphi_- \in \mathbb{Z}$ or the smallest integer larger than $\frac{-1}{2\pi}\cot\varphi_-$ if

$\frac{-1}{2\pi}\cot\varphi_- \notin \mathbb{Z}$. Then $\cos\varphi + 2\pi h \sin\varphi < 0$ holds for all $\varphi \in (\varphi_-, \varphi_+)$ if $h < h_-$. Hence, $\beta_h = 0$ if $h < h_-$ and we see that β necessarily is of the form

$$\beta(x) = \sum_{h=h_-}^{\infty} \beta_h e^{2\pi i h x} = \beta_{h_-} e^{2\pi i h_- x}\left\{1 + \sum_{h=h_-+1}^{\infty} \beta_{h_-}^{-1}\beta_h e^{2\pi i (h-h_-)x}\right\}$$

and this latter sum is exponentially small on S. So $\beta(x) = \beta_{h_-} e^{2\pi i h_- x}\big(1 + O(x^{-1})\big)$ as $x \to \infty$ in S.

From (1.5.17) we conclude that y might be singular at those points $x \in \mathbb{C}$ satisfying the equation $f_0(x) = \beta(x)e^{-x} f_1(x)$ and this equation can be rewritten as

$$x = \ln\big(\beta(x)\big) + \ln\big(f_1(x) f_0^{-1}(x)\big) + 2n\pi i, \quad n \in \mathbb{Z}. \tag{1.5.18}$$

In the following we restrict x to S, because there we know the asymptotic behaviour of f_0, f_1 and β. Observing that $\ln\big(\beta(x)\big) = \ln\beta_{h_-} + 2\pi i h_- x + O(x^{-1})$ and $\ln\big(f_1(x) f_0^{-1}(x)\big) = O(x^{-1})$, one may show (in a similar way as in the preceding example) that (1.5.18) has a unique solution $x = x_n$ for every $n \in \mathbb{N}$ large enough. As in the preceding example we obtain

$$x_n = \frac{\ln\beta_{h_-}}{1 - 2\pi i h_-} + \frac{2n\pi i}{1 - 2\pi i h_-} + O(n^{-1}), \quad n \in \mathbb{N}, \quad n \to \infty. \tag{1.5.19}$$

We conclude that the solution y of equation (1.5.12) might be singular at a distance at most $O(n^{-1})$ of $(1 - 2\pi i h_-)^{-1}\big(\ln\beta_{h_-} + 2n\pi i\big)$, as $n \in \mathbb{N}$, $n \to \infty$.

We will end this example with the remark that we are not really restricted to the upper half plane. The same study can be done in for example the lower half plane. In this thesis we will consider sectors containing the positive real axis. Of course this can also be achieved here by taking the Borel sum f_0 of $\hat{f}_0$ on $\{x \in \mathbb{C}^* \mid -\pi/2 - \theta_+ < \arg x < \pi/2 - \theta_-\}$, with θ_- and θ_+ two consecutive singular directions in the right half plane. In that case we can prove that if the 1-periodic function β has the property $\beta(x)e^{-x} \to 0$ as $x \to \infty$, then β necessarily must be a trigonometric polynomial (cf. [Bra01] for more precise results).